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RELATIVE ENTROPY OF ENTANGLEMENT FOR TWO-QUBIT STATE WITH z-DIRECTIONAL BLOCH VECTORS

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So far there is no closed formula for relative entropy of entanglement of arbitrary two-qubit states. In this paper we present a method, which guarantees the derivation of the relative entropy of entanglement for most states that have z-directional Bloch vectors. It is shown that the closest separable states for those states also have z-directional Bloch vectors though there are few exceptions.

Keywords: Relative entropy of entanglement; distillable entanglement.

1. Introduction

Research into entanglement of quantum states has a long history from the very beginning of quantum mechanics.[1,2] At that time the main motivation for the study of entanglement was to explore the non-local property of quantum mechanics. Still, this issue is not completely understood. Recent study on the entanglement is mainly due to its role as a physical resource for the various quantum information processing such as teleportation,[3] quantum cryptography,[4] and speed-up of quantum computer.[5]

In order to quantify how much a given quantum state is entangled, many entanglement measures were invented for last two decades. Among them the most important measure seems to be the distillable entanglement[6] because it measures how a given quantum state is useful in the real quantum information processing with overcoming the effect of the noises via the purification protocol. In spite of its importance the analytically derivational technique for it even in the relatively simple quantum system is not known. In fact, in order to compute the distillable entanglement, we should find an optimal purification protocol. However, it is a nontrivial problem to find the optimal protocol except very rare cases. For this reason many people tried to find more analytically tractable entanglement measures which may be able to provide information on the tight upper bound of the distillable entanglement. The representatives constructed in this reason are entanglement of formation (EOF)[6] and relative entropy of entanglement (REE).[7,8]

About a decade ago, Wootters[9] found how to compute the EOF for arbitrary two-qubit states. Although we still do not have closed formula of EOF for higher-dimensional quantum system, the Wootters' result has great impact in the study of entanglement. One example for an application of the Wootters' result is to examine the role of the quantum entanglement in a complex quantum system such as bio-system.[10] Another direction of application is to use the Wootters' result to find a truly multipartite entanglement measure. In this way, the three-tangle, measure for the genuine tripartite entanglement, was invented in Ref. 11.

On the contrary, we still do not have closed formula of the REE even for the two-qubit states.[12] In order to understand the distillable entanglement more profoundly, therefore, it is worthwhile to investigate the properties of the REE for the various two-qubit states. In this paper we would like to examine the REE for the states, which have z-directional Bloch vectors. We present three theorems in the following, which guarantees that the REE for most such states can be computed analytically or, at least, numerically.

The REE for state ρ is defined as

$$E_R(\rho) = \min_{\sigma\in\mathcal{D}} S(\rho\|\sigma) = \min_{\sigma\in\mathcal{D}} \mathrm{tr}[\rho\ln\rho - \rho\ln\sigma], \tag{1}$$

where $\mathcal{D}$ is a set of positive partial transpose (PPT) states. For various properties of the REE see Refs. 13–16. If our concern is restricted into the two-qubit state, it is possible to regard $\mathcal{D}$ as a set of the separable states, because there is no bound entangled state in the two-qubit Hilbert space. The separable state σ in Eq. (1) is called the closest separable state (CSS) of ρ. In order for the separable state σ to be CSS of some entangled states it should be edge state in the set $\mathcal{D}$, which means that the smallest eigenvalue of σ^Γ is zero, where the superscript Γ denotes partial transposition.[a]

2. Analysis

Although the definition of the REE is comparatively simple, the analytic computation of it is a highly difficult problem even for the most simple two-qubit case (see Chapter 8 of Ref. 12). Since the REE can be straightforwardly computed provided that the CSS is derived, this means that finding a CSS of the given entangled state is very difficult. Recently, however, the authors in Ref. 17 analyzed the converse procedure. When the edge separable state π is full-rank, they have presented a method for deriving the entangled state ρ, whose CSS is π. Still, however, finding a CSS for the arbitrary entangled state ρ is an unsolved problem.

[a] The converse of this statement, i.e. *if σ is an edge state in $\mathcal{D}$, there exist entangled states whose CSS are σ,* is not generally true.

In order to explore the issue for finding CSS or REE, authors in Ref. 18 approached the problem from the geometrical point of view. To explain the main results of Ref. 18 briefly it is convenient to express the given entangled state ρ in a form

$$\rho = \frac{1}{4}\left[I \otimes I + \boldsymbol{r}\cdot\boldsymbol{\sigma}\otimes I + I \otimes \boldsymbol{s}\cdot\boldsymbol{\sigma} + \sum_{i,j=1}^{3} g_{ij}\sigma_i \otimes \sigma_j\right] \tag{2}$$

where $\boldsymbol{\sigma}$ is usual Pauli matrices. The vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are Bloch vectors for each qubit and the tensor g_{ij} represents a correlation between qubits. Since appropriate local-unitary (LU) transformation for each qubit can make the correlation tensor g_{ij} to be diagonal, it is more convenient, without loss of generality, to express ρ as

$$\rho = \frac{1}{4}\left[I \otimes I + \boldsymbol{r}\cdot\boldsymbol{\sigma}\otimes I + I \otimes \boldsymbol{s}\cdot\boldsymbol{\sigma} + \sum_{n=1}^{3} g_{n}\sigma_n \otimes \sigma_n\right]. \tag{3}$$

For example, for the four Bell states

$$\begin{aligned} |\beta_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\beta_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\beta_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad |\beta_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned} \tag{4}$$

the Bloch vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are vanishing and the corresponding correlation vectors become

$$\boldsymbol{g}_1 = (1,-1,1) \quad \boldsymbol{g}_2 = (-1,1,1) \quad \boldsymbol{g}_3 = (1,1,-1) \quad \boldsymbol{g}_4 = (-1,-1,-1). \tag{5}$$

In Ref. 18 it was shown that if ρ is one of Bell-diagonal, generalized Vedral-Plenio (VP) and generalized Horodecki states, its CSS is

$$\pi = \frac{1}{4}\left[I \otimes I + \boldsymbol{r}\cdot\boldsymbol{\sigma}\otimes I + I \otimes \boldsymbol{s}\cdot\boldsymbol{\sigma} + \sum_{n=1}^{3} \gamma_{n}\sigma_n \otimes \sigma_n\right]. \tag{6}$$

The correlation vector of π, $\boldsymbol{\gamma}$, can be computed from a fact that the straight line in the correlation vector space, which connects $\boldsymbol{\gamma} = (\gamma_x, \gamma_y, \gamma_z)$ and $\boldsymbol{g} = (g_x, g_y, g_z)$ passes through one of Eq. (5), which is the nearest one from $\boldsymbol{g}$. Since this fact with the edge state criterion uniquely determines the correlation vector $\boldsymbol{\gamma}$ of the CSS, it is straightforward to compute the REE for the Bell-diagonal, generalized VP and generalized Horodecki states. For example, let us choose the Bell-diagonal, VP and Horodecki states as follows:

$$\begin{aligned} \rho_B &= \lambda_1|\beta_1\rangle\langle\beta_1| + \lambda_2|\beta_2\rangle\langle\beta_2| + \lambda_3|\beta_3\rangle\langle\beta_3| + \lambda_4|\beta_4\rangle\langle\beta_4| \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\max(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \lambda_3) \\ \rho_{vp} &= \lambda_1|\beta_3\rangle\langle\beta_3| + \lambda_2|01\rangle\langle 01| + \lambda_3|10\rangle\langle 10| \\ \rho_H &= \lambda_1|\beta_3\rangle\langle\beta_3| + \lambda_2|00\rangle\langle 00| + \lambda_3|11\rangle\langle 11|. \end{aligned} \tag{7}$$

Following Ref. 18 it is easy to show that the corresponding CSS for these states are

$$\begin{aligned}
\pi_B &= \frac{\lambda_1}{2(1-\lambda_3)}|\beta_1\rangle\langle\beta_1| + \frac{\lambda_2}{2(1-\lambda_3)}|\beta_2\rangle\langle\beta_2| + \frac{1}{2}|\beta_3\rangle\langle\beta_3| + \frac{\lambda_4}{2(1-\lambda_3)}|\beta_4\rangle\langle\beta_4| \\
\pi_{vp} &= \left(\frac{\lambda_1}{2}+\lambda_2\right)|01\rangle\langle 01| + \left(\frac{\lambda_1}{2}+\lambda_3\right)|10\rangle\langle 10| \\
\pi_H &= \frac{(\lambda_1+2\lambda_2)(\lambda_1+2\lambda_3)}{2}|\beta_3\rangle\langle\beta_3| + \frac{(\lambda_1+2\lambda_2)^2}{4}|00\rangle\langle 00| + \frac{(\lambda_1+2\lambda_3)^2}{4}|11\rangle\langle 11|
\end{aligned} \tag{8}$$

and their REE become

$$\begin{aligned}
E_r(\rho_B) &= -H(\lambda_3) + \ln 2 \\
E_r(\rho_{vp}) &= H\left(\frac{\lambda_1}{2}+\lambda_2\right) - H(\Lambda) \quad \left(\Lambda = \frac{1}{2}\left[1+\sqrt{\lambda_1^2+(\lambda_2-\lambda_3)^2}\right]\right) \\
E_r(\rho_H) &= \lambda_1\ln\lambda_1 + \lambda_2\ln\lambda_2 + \lambda_3\ln\lambda_3 + 2H\left(\frac{\lambda_1}{2}+\lambda_2\right) - \lambda_1\ln 2
\end{aligned} \tag{9}$$

where $H(p) \equiv -p\ln p - (1-p)\ln(1-p)$. It is worthwhile noting that $E_r(\rho_{vp})$ and $E_r(\rho_H)$ are invariant under the exchange of λ_2 and λ_3. In fact, one can conjecture this symmetry from the physical point of view.

In this paper we would like to examine the REE for the two qubit states, whose Bloch vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are z-directional. Thus, we assume $\boldsymbol{r} = (0,0,r)$ and $\boldsymbol{s} = (0,0,s)$. For more simplicity we assume that the first two components of the correlation vector $\boldsymbol{g}$ are identical, i.e. $g_x = g_y$. Then, the quantum state ρ can be written as

$$\rho = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & De^{i\varphi} & 0 \\ 0 & De^{-i\varphi} & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \tag{10}$$

where

$$\begin{gathered}
A_1 = \frac{1+r+s+g_z}{4}, \quad A_2 = \frac{1+r-s-g_z}{4}, \quad A_3 = \frac{1-r+s-g_z}{4} \\
A_4 = \frac{1-r-s+g_z}{4}, \quad D = \frac{g_x}{2\cos\varphi} \geq 0.
\end{gathered} \tag{11}$$

We also impose

$$D^2 > A_1 A_4 \tag{12}$$

to require that ρ is an entangled state.

Now we conjecture that the CSS of ρ is of a form

$$\pi = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & ye^{i\varphi} & 0 \\ 0 & ye^{-i\varphi} & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix} \tag{13}$$

with $y = \sqrt{r_1 r_4} \leq \sqrt{r_2 r_3}$. In the following we will show that most entangled states of the form (10) have really their CSS as the form (13). However, for extremely asymmetric states we will show that our conjecture is not true.

If π is really the CSS of ρ, the following coupled equations should be satisfied[17]:

$$r_1 - x\frac{r_1 r_4}{r_1 + r_4} = A_1 \tag{14a}$$

$$r_4 - x\frac{r_1 r_4}{r_1 + r_4} = A_4 \tag{14b}$$

$$r_2 + x\frac{2r_1 r_4}{(r_1 + r_4)z^2\ell}[2r_1 r_4 \ell + (r_2 - r_3)(r_2\ell - z)] = A_2 \tag{14c}$$

$$r_3 + x\frac{2r_1 r_4}{(r_1 + r_4)z^2\ell}[2r_1 r_4 \ell - (r_2 - r_3)(r_3\ell - z)] = A_3 \tag{14d}$$

$$y + x\frac{y}{(r_1 + r_4)z^2\ell}[2r_1 r_4(r_2 + r_3)\ell + (r_2 - r_3)^2 z] = D, \tag{14e}$$

where x is a positive parameter and

$$z = \sqrt{(r_2 - r_3)^2 + 4r_1 r_4} \quad \ell = \ln\frac{r_2 + r_3 + z}{r_2 + r_3 - z}. \tag{15}$$

In this case one can show after tedious calculation that the REE of ρ becomes

$$\begin{aligned} E_r(\rho) &\equiv \mathrm{tr}(\rho\ln\rho) - \mathrm{tr}(\rho\ln\pi) \\ &= (A_1\ln A_1 + A_4\ln A_4 + A_+\ln A_+ + A_-\ln A_-) \\ &\quad - \Bigg(A_1\ln r_1 + A_4\ln r_4 + \frac{A_2 + A_3}{2}\ln(r_2 r_3 - r_1 r_4) \\ &\quad + \frac{(A_2 - A_3)(r_2 - r_3) + 4Dy}{2z\ell^{-1}}\Bigg) \end{aligned} \tag{16}$$

where

$$A_\pm = \frac{1}{2}\left[(A - 2 + A_3) \pm \sqrt{(A_2 - A_3)^2 + 4D^2}\right]. \tag{17}$$

Now we present the following three theorems, which provide the REE and CSS of the entangled state ρ given in Eq. (10).

Theorem 1. *If* $A_1 = A_4 = 0$, $E_r(\rho)$ *becomes*

$$E_r(\rho) = H(A_2) - H(A_+).$$

Proof. If $A_1 = A_4 = 0$, Eqs. (14a) and (14b) give solutions $r_1 = r_4 = \epsilon$, where ϵ is an infinitesimal positive parameter, which will be taken to be zero after calculation. Then, the remaining equations in Eq. (14) eventually generate the following solutions:

$$r_2 = A_2, \quad r_3 = A_3, \quad x = \frac{2D}{|A_2 - A_3|}, \ln \frac{\max(A_2, A_3)}{\min(A_2, A_3)}. \tag{18}$$

Therefore, CSS π in this case is

$$\pi = A_2|01\rangle\langle 01| + A_3|10\rangle\langle 10|. \tag{19}$$

Making use of Eq. (16) it is straightforward to compute the REE, which completes the proof.

As an example of Theorem 1 let us consider

$$\rho = p|\psi\rangle\langle\psi| + q_1|01\rangle\langle 01| + q_2|10\rangle\langle 10| \tag{20}$$

where $p + q_1 + q_2 = 1$ and $|\psi\rangle = \alpha|01\rangle + \beta|10\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$). Then the CSS of ρ is

$$\pi = (p|\alpha|^2 + q_1)|01\rangle\langle 01| + (p|\beta|^2 + q_2)|10\rangle\langle 10| \tag{21}$$

and the corresponding REE is

$$E_r(\rho) = H(p|\alpha|^2 + q_1) - H(A_+) \tag{22}$$

where

$$A_\pm = \frac{1}{2}\left[1 \pm \sqrt{p^2 + (q_1 - q_2)\{2p(|\alpha|^2 - |\beta|^2) + (q_1 - q_2)\}}\right]. \tag{23}$$

When $\alpha = \beta = 1/\sqrt{2}$, it is easy to show that Eq. (22) reduces to the second equation of Eq. (9) when $\lambda_1 = p$, $\lambda_2 = q_1$ and $\lambda_3 = q_3$. □

Theorem 2. *If both A_1 and A_4 are not zero, and $A_2 = A_3$, the REE of ρ becomes*

$$E_r(\rho) = \Omega_1 - \Omega_2 \tag{24}$$

where

$$\begin{aligned} \Omega_1 &= A_1 \ln A_1 + A_4 \ln A_4 + (A_2 + D)\ln(A_2 + D) + (A_2 - D)\ln(A_2 - D) \\ \Omega_2 &= A_1 \ln r_1 + A_4 \ln r_4 + A_2 \ln(r_2^2 - r_1 r_4) + D \ln \frac{r_2 + y}{r_2 - y}. \end{aligned} \tag{25}$$

In Eq. (25)

$$\begin{aligned} r_1 &= \frac{1}{F}[2A_1(A_1 + A_2)(A_1 + A_2 + A_4) - D^2(A_1 - A_4) + \Delta] \\ r_4 &= \frac{1}{F}[2A_4(A_2 + A_4)(A_1 + A_2 + A_4) + D^2(A_1 - A_4) + \Delta] \\ r_2 &= \frac{1}{F}[2(A_1 + A_2)(A_2 + A_4)(A_1 + A_2 + A_4) - D^2(A_1 + 2A_2 + A_4) - \Delta] \end{aligned} \tag{26}$$

where $y = \sqrt{r_1 r_4}$ *and*

$$\begin{aligned} F &= 2(A_1 + A_2 + A_4 + D)(A_1 + A_2 + A_4 - D) \\ \Delta &= D\sqrt{D^2(A_1 - A_4)^2 + 4A_1A_4(A_1 + A_2)(A_2 + A_4)}. \end{aligned} \tag{27}$$

Remark: Under $A_1 \leftrightarrow A_4$, r_2 is invariant and, r_1 and r_4 are changed into each other. This fact indicates that $E_r(\rho)$ is invariant under $A_1 \leftrightarrow A_4$. The appearance of this symmetry is plausible from the physical point of view.

Proof. Since both A_1 and A_4 are not zero, Eqs. (14a) and (14b) enable us to express r_4 and x in terms of r_1 as follows:

$$r_4 = r_1 - (A_1 - A_4), \quad x = \frac{(r_1 - A_1)(r_1 + r_4)}{r_1 r_4}. \tag{28}$$

Since $A_2 = A_3$, Eqs. (14c) and (14d) imply $r_2 = r_3$. Then inserting $r_2 = r_3$ and Eq. (28) into Eq. (14c), one can express r_2 in terms of r_1 as follows:

$$r_2 = -r_1 + (A_1 + A_2). \tag{29}$$

In fact, Eq. (29) can be derived from a normalization $r_1 + 2r_2 + r_4 = 1$. Finally, we consider Eq. (14e), which reduces to

$$y^2 - Dy + r_2(r_1 - A_1) = 0. \tag{30}$$

Thus, one can express y in terms of r_1 as a form

$$y = \frac{1}{2}\left[D \pm \sqrt{D^2 - 4r_2(r_1 - A_1)}\right]. \tag{31}$$

Since $y^2 = r_1 r_4$, one can compute r_1 from Eq. (31), which is

$$r_1 = \frac{1}{F}[2A_1(A_1 + A_2)(A_1 + A_2 + A_4) - D^2(A_1 - A_4) \pm \Delta]. \tag{32}$$

Therefore, one can easily compute r_2 and r_4 by making use of Eqs. (28) and (29). The undetermined sign can be fixed by Eq. (30). Then, Eq. (16) completes a proof of Theorem 2.

As an example of Theorem 2, let us consider

$$\rho = p_1|\beta_3\rangle\langle\beta_3| + p_2|\beta_4\rangle\langle\beta_4| + q_1|00\rangle\langle 00| + q_2|11\rangle\langle 11| \tag{33}$$

with $p_1 + p_2 + q_1 + q_2 = 1$. Then, it is straightforward to show

$$\begin{aligned} r_1 &= \frac{2q_1(p_1 + p_2 + 2q_1)(p_1 + p_2 + 2q_1 + 2q_2) - (p_1 - p_2)^2(q_1 - q_2) + 4\Delta}{8(p_1 + q_1 + q_2)(p_2 + q_1 + q_2)} \\ r_2 &= \frac{(p_1 + p_2 + 2q_1)(p_1 + p_2 + 2q_2)(p_1 + p_2 + 2q_1 + 2q_2) - (p_1 - p_2)^2 - 4\Delta}{8(p_1 + q_1 + q_2)(p_2 + q_1 + q_2)} \end{aligned} \tag{34}$$

where

$$\Delta = \frac{p_1 - p_2}{4}\sqrt{4q_1q_2(p_1+p_2+2q_1)(p_1+p_2+2q_2)+(p_1-p_2)^2(q_1-q_2)^2} \tag{35}$$

and r_4 is obtained from r_1 by exchanging q_1 and q_2. Then it is easy to compute the REE of ρ by making use of Theorem 2. When $p_2 = 0$, it is also straightforward to show that the REE of ρ reduces to third equation of Eq. (9) if one identifies $\lambda_1 = p_1$, $\lambda_2 = q_1$ and $\lambda_3 = q_2$. □

Theorem 3. *For other cases the CSS of ρ can be obtained by solving an equation*

$$\frac{r_2 + r_3 + z}{r_2 + r_3 - z} = \exp\left[\frac{z(r_1 - A_1)(r_2 - r_3)^2}{y(D-y)z^2 - 2r_1r_4(r_1 - A_1)(r_2 + r_3)}\right] \tag{36}$$

where

$$\begin{aligned} r_4 &= r_1 - (A_1 - A_4) \\ r_2 &= \frac{1}{4}[(4A_1 + 3A_2 + A_3) - 4r_1 + \sqrt{\Gamma}] \\ r_3 &= \frac{1}{4}[(4A_1 + A_2 + 3A_3) - 4r_1 - \sqrt{\Gamma}] \end{aligned} \tag{37}$$

and

$$\begin{aligned} \Gamma = {} & 16D\sqrt{r_1r_4} - 8(2A_1 + A_2 + A_3 + 2A_4)r_1 \\ & + [(A_2 - A_3)^2 + 8A_1(2A_1 + A_2 + A_3)]. \end{aligned} \tag{38}$$

Remark 1. If Eqs. (37) and (38) are used, one can make the LHS and RHS of Eq. (36) in terms of r_1 only. Thus, Eq. (36) is an equation with only one variable, which can be solved analytically or numerically.

Remark 2. If Eq. (37) does not provide a solution for some entangled state ρ, this fact indicates that the CSS of ρ is not of the form (13). In this case, therefore, CSS of ρ seems to have different structure from ρ.

Proof. From Eqs. (14a) and (14b) one can express r_4 and x in terms of r_1, which is exactly the same with Eq. (28). The remaining equations in Eq. (14) reduce to

$$2z(r_1 - A_1)(r_2 - r_3) = \ell\big[(r_2 - A_2)z^2 + 2(r_1 - A_1)\{r_2(r_2 - r_3) + 2r_1r_4\}\big] \tag{39a}$$

$$2z(r_1 - A_1)(r_2 - r_3) = \ell\big[(A_3 - r_3)z^2 + 2(r_1 - A_1)\{r_3(r_2 - r_3) - 2r_1r_4\}\big] \tag{39b}$$

$$2z(r_1 - A_1)(r_2 - r_3) = \ell\,\frac{2y(D-y)z^2 - 4r_1r_4(r_1 - A_1)(r_2 + r_3)}{r_2 - r_3}. \tag{39c}$$

Since the LHS of Eq. (39) are all identical, the RHS of them should be equal. By equalizing the RHS of Eq. (39a) with the RHS of Eq. (39c) one can derive

$$(r_2 - r_3)(A_2 - r_2) + 2y(D-y) - 2r_2(r_1 - A_1) = 0. \tag{40}$$

Similarly, one can derive

$$(r_2 - r_3)(A_3 - r_3) - 2y(D - y) + 2r_3(r_1 - A_1) = 0 \tag{41}$$

from Eqs. (39b) and (39c). Adding Eqs. (40) and (41) one can express $r_2 + r_3$ in terms of r_1 in a form

$$r_2 + r_3 = 1 - r_1 - r_4. \tag{42}$$

In fact, Eq. (42) is a normalization for the CSS π. Combining Eqs. (41) and (42) one can make the following second degree equation

$$\begin{aligned} &2r_3^2 + [4r_1 - (4A_1 + A_2 + 3A_3)]r_3 \\ &\quad + [A_3\{(2A_1 + A_2 + A_3) - 2r_1\} - 2y(D - y)] = 0, \end{aligned} \tag{43}$$

which has roots

$$r_3 = \frac{1}{4}[(4A_1 + A_2 + 3A_3) - 4r_1 \pm \sqrt{\Gamma}]. \tag{44}$$

Inserting Eq. (44) into Eq. (42), one can express r_2 in terms of r_1 as a form

$$r_2 = \frac{1}{4}[(4A_1 + 3A_2 + A_3) - 4r_1 \mp \sqrt{\Gamma}]. \tag{45}$$

The undetermined sign in Eqs. (44) and (45) can be fixed by Eq. (40). Finally, the parameter r_1 is determined by Eq. (39c), which reduces to Eq. (36). This completes a proof. □

As an example of Theorem 3, let us re-consider the model which was considered by Rains in Ref. 19, where the entangled state is

$$\rho = \begin{pmatrix} \frac{1}{12} & 0 & 0 & 0 \\ 0 & \frac{45907}{90000} - \frac{7\xi}{150} & \frac{1201}{3750} + \frac{49\xi}{3600} & 0 \\ 0 & \frac{1201}{3750} + \frac{49\xi}{3600} & \frac{29093}{90000} + \frac{7\xi}{150} & 0 \\ 0 & 0 & 0 & \frac{1}{12} \end{pmatrix} \tag{46}$$

with $\xi = 1/\ln(73/23)$. Then Eq. (36) directly gives $r_1 = 1/6$ and the resulting CSS of ρ is

$$\pi = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{55}{144} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{41}{144} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}. \tag{47}$$

This is in agreement with Rains' result.

As a second example of Theorem 3 let us consider

$$\rho = p|\beta_3\rangle\langle\beta_3| + q_1|01\rangle\langle 01| + q_2|10\rangle\langle 10| + q_3|00\rangle\langle 00| + q_4|11\rangle\langle 11| \tag{48}$$

with $p = 0.66$, $q_1 = 0.16$, $q_2 = 0.03$, $q_3 = 0.06$ and $q_4 = 0.09$. Then, Eq. (36) cannot be solved analytically. The numerical calculation shows that the CSS is

$$\pi = p'|\beta_3\rangle\langle\beta_3| + q_1'|01\rangle\langle 01| + q_2'|10\rangle\langle 10| + q_3'|00\rangle|\langle 00| + q_4'|11\rangle\langle 11| \tag{49}$$

where $p' = 0.306933$, $q_1' = 0.252429$, $q_2' = 0.132241$, $q_3' = 0.139198$ and $q_4' = 0.169198$.

Numerical calculation shows that most entangled states of the form (10) have their CSS as a form of (13). However, there are states whose CSS are not of the form (13). For example, the state (48) with $p = 0.66$, $q_1 = 0.05$, $q_2 = 0.07$, $q_3 = 0.04$ and $q_4 = 0.18$ does not have CSS of the form (13). It seems to be interesting to derive a criterion that clarifies which entangled states ρ do not have CSS of the form (13).

3. Discussion

We have assumed *ab initio* that the Bloch vectors of ρ are z-directional. In addition, we have assumed that the first two components of the correlation vector are equal. These assumptions are chosen only for simplicity. In the near future, we would like to re-visit the REE problem for two-qubit states by removing these assumptions as much as possible. This may shed light on the explicit derivation for the closed formula of REE in the two-qubit system.

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